# DEADLOCK-FREE CONDITIONS FOR A CLASS OF PETRI-NETS(Mathematical Theory of Control and Systems)

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## ABSTRACT

Deadlock-freeness (liveness) of a structually restricted subclass of Petri nets, Extened Marked Graphs (EMG), is considered in this paper. EMG can be also viewed as augmented marked graphs with some specified arcs that express a permissive control function of places on firing of corresponding transitions. Necessary and sufficient conditions for the liveness of EMG are derived in terms of the initial token distribution and the net structure.

## 1. INTRODUCTION

A large class of concurrent discrete event systems can be modelled by Petri nets and control problems of such systems are basically reduced to solving reachability and liveness on the nets.

It is known that liveness and reachability in Petri nets are equivalent with respect to decidability [1]. Recently reachability problem has been settled by Mayr [2]; he has presented an algorithm which terminates in finite steps and determines whether for given $M_0$, $M \varepsilon N^n$, M is reachable from $M_0$ or not. Thus equivalent problems such as liveness, coverability and submarking reachability are decidable in this sense. On the otherhand it is also known that the decision algorithm requires in general an exponential amount of space and steps with respect to the size of the net [3]. Thus it may not be applicable to the analysis and synthesis problems encountered in practical concurrent systems.

As structually restricted Petri nets, marked graphs possess definite relationships between the net structure and dynamic properties of the system being modelled. The reachability problem and related problems, e.g., liveness, safe-liveness and submarking reachability were successfully resolved for marked graphs to obtain necessary and sufficient conditions on the net structure and the initial token distributions that ensure these properties [4, 5, 6]. Marked graphs, however, have their natural drawback of weak modeling ability. For example, conditional branching function cannot be modelled with marked graphs.

In this paper, Extended Marked Graphs (EMG) are defined as the augmented marked graphs by adding some specific arcs between some pairs of places and transitions to represent the permissive controlling function of a place for firings of the corresponding transitions.
If a token capacity is prescribed for each controlling place, inhibitive function (zero testing capability) [7] can also be modelled by EMG.

The purpose of this paper is to investigate structural properties of EMG with respect to the dynamics of the token distributions and to derive necessary and sufficient conditions for liveness of EMG in terms of the net structure and the initial token distributions.

## 2. DEFINITIONS AND NOTATIONS

The structure of Petri nets can be defined as a directed bipertite graph with two disjoint sets of nodes, called a set S of places ( symbol: O) and a set T of transitions (symbol: |).Marked graphs are a subclass of Petri nets, where each place has exactly one incoming arc and exactly one outgoing arc. In the following we do not consider Petri nets with multiple arcs. Assume that cardinality of S is n. To each place s, we associate an non-negative integer M(s), called a number of tokens on s. Marking M $\varepsilon$ $N^n$ is then defined as an non-negative integral vector whose component M(s) equals the number of tokens of place s. For a subset D of S, M(D) denotes a subvector of M with component M(s), s $\varepsilon$ D. $\Sigma$M(D) denotes the sum of tokens of places in D. $^{\bullet}D$ denotes the set of all transitions t such that there exists an arc e:t $\rightarrow$ s, s $\varepsilon$ D. $^{\bullet}D$ is called the set of input transitions of D. Similarly, $D^{\bullet}$ denotes the set of all transitions t such that there exists an arc e:s $\rightarrow$ t, s $\varepsilon$ D and is called the set of output transitions of D. For a subset Q of T, the set of input places $^{\bullet}Q$ and the set of output places $Q^{\bullet}$ of Q are similarly defined. Subset D of places is called a deadlock iff $^{\bullet}D \subseteq D^{\bullet}$. D is called a trap iff $D^{\bullet} \subseteq {}^{\bullet}D$. Deadlock D (Trap E) is said to be minimal iff no subset of D (E) is a deadlock (trap), respectively. Note that for marked graphs the definitions of minimal deadlock and trap coincide and are equivalent to a set of places on a directed circuit [1]. Now the dynamic behavior of Petri nets is stipulated by the following simple firing axiom. For a transition t, t is said to be firable at M iff $M(s) > 0$ for each $s \in {}^{\cdot}t$. A firing of firable transition t at M is said to be legal and consists of the following change of the tokens. The resulting new marking $M' \geq 0$ is defined as

$$M'(s) = M(s) + 1,\ s \in t^{\cdot} \wedge s \notin {}^{\cdot}t \qquad (1)$$

$$M'(s) = M(s) - 1,\ s \in {}^{\cdot}t \wedge s \notin t^{\cdot} \qquad (2)$$

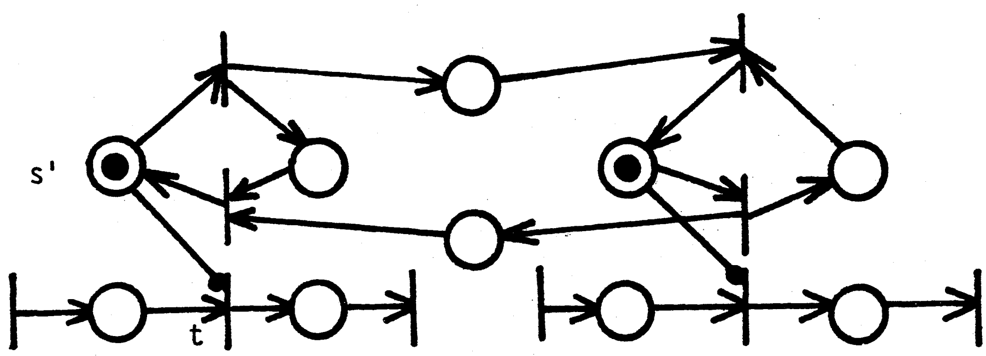
$$M'(s) = M(s) \qquad ,\ \text{otherwise.} \qquad (3)$$

If there exists a legal sequence of firings that brings $M_0$ to M, M is said to be reachable from $M_0$. $R(M_0)$ denotes a set of all markings reachable from $M_0$ and is called the reachability set of $M_0$. For a transition t, t is said to be live at $M_0$ iff for any $M \in R(M_0)$ there exists $M' \in R(M)$ such that t is firable at $M'$. If there exists $M \in R(M_0)$ such that t is not firable at any $M' \in R(M)$ then t is said to be dead at M. If each t of T is live at $M_0$, then the Petri net is said to be live or, equivalently, deadlock-free. It is known that the reachability problem, i.e., to decide for any given $M_0$, $M \in N^n$ whether $M \in R(M_0)$ or not, is equivalent to the liveness problem, i.e., to decide for a given $M_0 \in N^n$ whether the Petri net is live or not. For marked graphs the following results are well known [8].

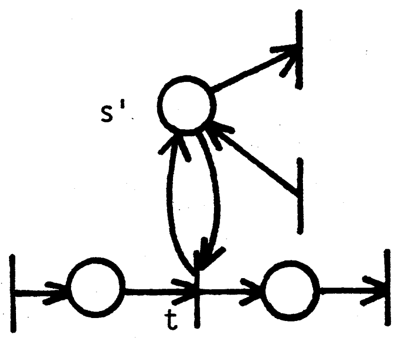
(i) Token sum of places on any directed circuit is invariant through any firing sequence.

(ii) A marked graph is live at $M_0$ iff there exists no unmarked directed circuit at $M_0$.

Given a marked graph, we can augment it by adding some special arcs, say, $s' \multimap t$ as shown in Fig. 1 (a). The firability condition of t is defined such that t is firable at M iff M(s) > 0 for each $s \in {}^{\bullet}t$, and in addition $M(s') > 0$ for each $s' : s' \multimap t$. The transition rules (1), (2) and (3) of markings still apply here except that $M(s')$ remains unchanged when t fires. Hereafter we mean by a directed path ( circuit ) in EMG a directed path ( circuit ) of directed arcs $\rightarrow$ in EMG.



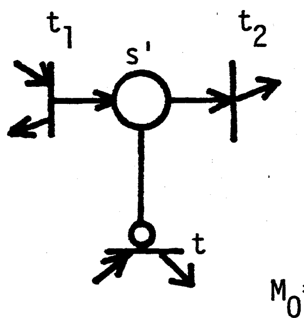

(a) a traffic system modelled by EMG



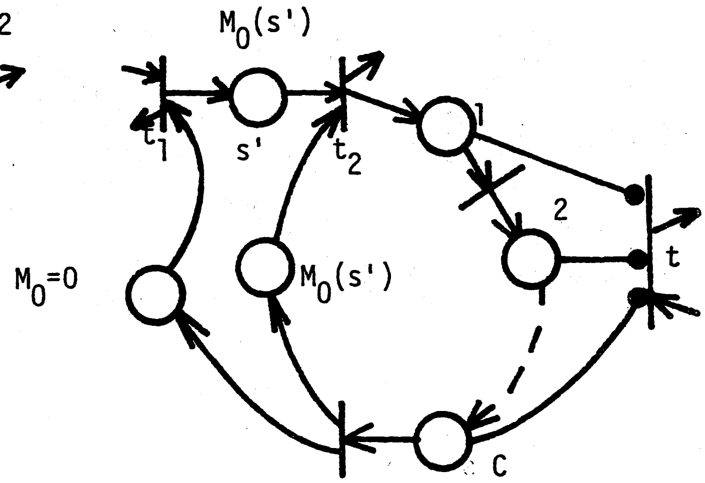

(b) Petri net equivalent

Fig. 1 Extended Marked Graph

The added arc $s' \multimap t$ implies a permissive controlling function of place $s'$ on the firing of t in that only while $s'$ is marked the firing of t is allowed and the firing does not affect the marking of $s'$. $s'$ is called a controlling place of t. The resulting augmented marked graph is called an Extended Marked Graph ( EMG ). As is easily seen, the arc $s' \multimap t$ is equivalent to $s' \rightleftarrows t$ in a Petri net model (see Fig. 1 (b)). Thus EMG are a subclass of Petri nets. The inhibitor $s' \multimap t$ introduced in [7] is an arc such that t may be fired only while $s'$ is unmarked. Note that it can be modelled by EMG if $s'$ has a prescribed token capacity C. ( See Fig. 2.) In the following section some structural properties of EMG are investigated to derive necessary and sufficient conditions for liveness.




(a) an inhibitor




(b) EMG model

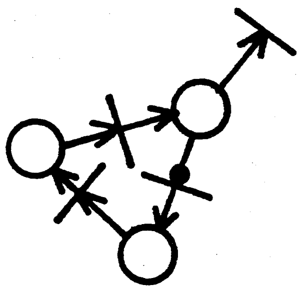
$M_0(1) + \cdots + M_0(C) = C - M_0(s')$

Fig. 2 inhibitor modelled by EMG

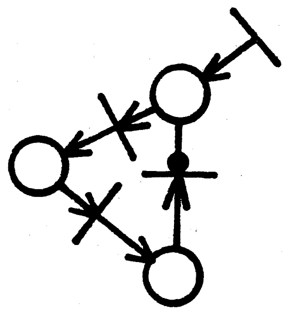
## 3. THE LIVENESS OF EMG

To investigate a relation between the dynamic behavior and the net structure of Petri nets, deadlock and trap defined in the previous section play a central role. These concept are generalized by using linear algebra in [9] but their topological interpretation are not clarified.

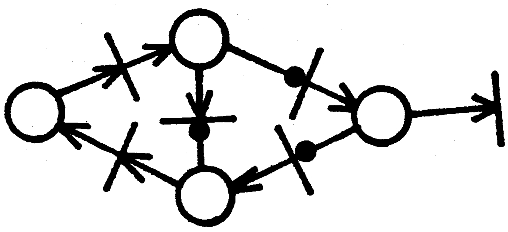
We begin with by considering possible structures of minimal deadlock and trap of EMG. A directed circuit in EMG is a minimal deadlock and also a minimal trap. Another possible structure of minimal deadlock and minimal trap can be shown as depicted in Fig. 3. These are only structures possible for a minimal deadlock and trap in EMG. Note that a controlling arc —● should be interpreted as two way arcs ⇄.



(a) deadlock



(b) trap



(c) a deadlock containing a trap

Fig. 3 a minimal deadlock and trap

For any deadlock D of a Petri net, if $\Sigma M(D) = 0$ at some $M \in R(M_0)$ then $\Sigma M'(D) = 0$ at any $M' \in R(M)$. Also for any trap E, if $\Sigma M(E) \neq 0$ at some $M \in R(M_0)$ then $\Sigma M'(E) \neq 0$ at any $M' \in R(M)$. Thus if there exists a deadlock D and M such that $\Sigma M(D) = 0$ then any transition of $D^{\bullet}$ is dead at M. It is then clear that non-existence of unmarked deadlock is necessary for liveness. It is not in general sufficient for liveness, i.e., there exists a non-live Petri net such that each deadlock cannot become unmarked through any possible firing sequence. We impose the following structural constraint on EMG for the condition to be necessary and sufficient.

Assumption 1. For each transition t, if there exists a directed circuit passing through more than two controlling places of t, then the token sum of the circuit is greater than or equal to the number of controlling places.

Let $Dead_M$ [1] denote the maximum set of dead transitions with respect to possible reachable markings from $M_0$, i.e.,

$$Dead_M := \sup_{M' \in R(M_0)} \{ t \in T \mid t \text{ is dead at } M' \} \quad (4)$$

Suffix M denotes a marking which attains the righthand set of (4).

Lemma 1. For any $t \in Dead_M$, there exists a $s \in {}^{\bullet}t$ such that $M'(s) = 0$ for any $M' \in R(M)$.

Proof: It is known [1] that for a subset D of S, $|D| \geqq 2$, if each $s \in D$ has at least one live input transition and M(D) will never be positive then there exists a strongly connected state-machine ( SCSM- ) component including at least two places of D. In EMG, SCSMcomponent is equivalent to a directed circuit. The proof is shown by contradiction. Suppose the contrary. Then we can show as in [1] that for some $t \in Dead_M$, a set L,

$$L := \{ s \in {}^{\bullet}t \mid s \text{ has at least one live input transition} \}$$

satisfies $|L| \geqq 2$. Suppose $|L| = 2$ without loss of generality. It is clear that M(L) will never be positive. Thus there exists a directed circuit including L and the places of L are controlling places of t. By Assumption 1 this directed circuit has a token sum greater than or equal two and any transition on the circuit must be live. Then M'(L) becomes positive at some $M' \in R(M)$. This leads to the desired contradiction.

By Lemma 1, we obtain the following result.

Theorem 1. Under Assumption 1, EMG is live at $M_0$ iff there exists no unmarked deadlock at any M reachable from $M_0$.

Proof: Only the sufficient part is proved. Suppose EMG is not live. Then $Dead_M$ is not empty. Let D be a set of places such that

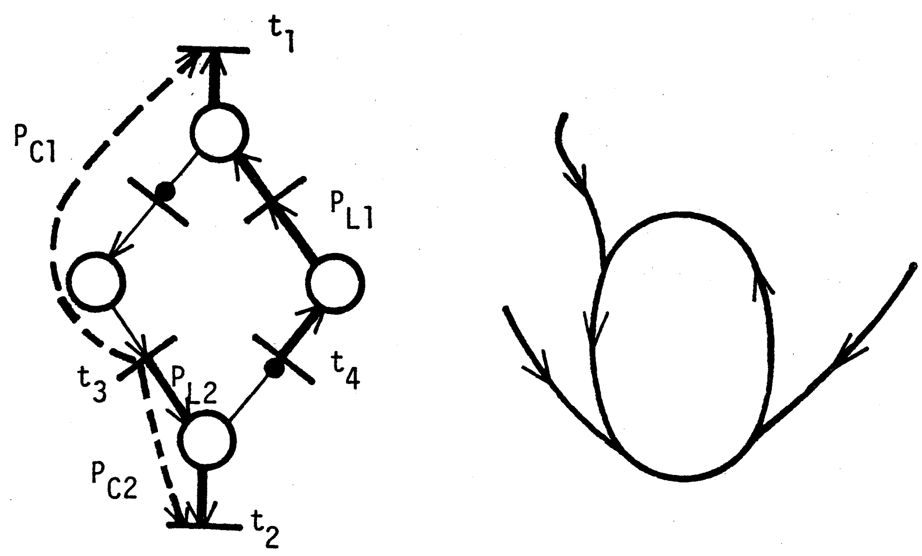
$$D := \{ s \in {}^{\bullet}( Dead_M ) \mid M'(s) = 0 \text{ for any } M' \in R(M) \}.$$

By Lemma 1, D is well defined. From the maximality of $Dead_M$, any $t \in T - Dead_M$ is live. Thus any $t \in {}^{\bullet}s$, $s \in D$, cannot be in $T - Dead_M$, i.e., ${}^{\bullet}D \subseteq Dead_M = D^{\bullet}$. This implies D is an unmarked deadlock.

Contrary to the invariant property of circuit token count in marked graphs, $\Sigma M(D)$ is not invariant for a deadlock in EMG. It remains to consider the conditions for a deadlock to preserve the positiveness of the token count throughout all possible firing sequences. It suffices to consider the conditions for a minimal deadlock D as depicted in Fig. 3. In case D contains a trap D' as in Fig. 3 (c), if $M_0(D') \neq 0$ then D cannot become unmarked from the property of trap D'. Such D is said to satisfy trap-condition at $M_0$. On practical points of view, we may simplify the cases so that a deadlock does not satisfy trap-condition iff it contains no trap. Consider a minimal deadlock D which contains no trap as shown in Fig. 4 (a). A transition t is called an outlet transition of D if t is not an input transition but an output transition of D. $t_1$ and $t_2$ are outlet transitions in Fig. 4. A directed path from some transition of D to an outlet transition t which does not pass through D is called a self-controlling path of t and denoted by $P_C$. A starting transition of $P_C$ is called a branching transition. In Fig. 4, paths $t_3 \to \cdots \to t_1$ and $t_3 \to \cdots \to t_2$ are $P_C$'s of $t_1$ and $t_2$, respectively, and $t_3$ is a branching transition. Without loss of generality, we assume that there exists at most one branching transition between two consequtive controlling places. A directed path in D from a branching transition to the nearest outlet transition is called a leading path and is denoted by $P_L$. If there exists no branching transition in the section, $P_L$ is defined from the first controlled transition to the nearest outlet transition. In Fig. 4, path $t_4 \to \cdots \to t_1$ and $t_3 \to \cdots \to t_2$ in D are $P_L$'s. Now soppose that EMG is live and contains a minimal deadlock D which does not satisfy the trap-condition. If all outlet transitions have no self-controlling path, then D becomes unmarked by the firings of the outlet transitions. We restrict the structure of EMG by making the following assumption.

Assumption 2. For each minimal deadlock, each outlet transition has at most one self-controlling path.

For a minimal deadlock D, consider a subnet composed of places and transitions on all sets of $P_C$ and $P_L$ and denote it as { $P_C$,$P_L$ }. From Assumption 2, the structure of each connected component of { $P_C$, $P_L$} is either the one as depicted in Fig. 4 (b) or simple path. A structure of Fig. 4 (b) is called a dendroid R of D. In Fig. 4 (a), { $P_{L1}$, $P_{C1}$, $P_{L2}$, $P_{C2}$ } constitutes a dendroid. Here the direction is picked so as to coincide with the direction of $P_L$.




(a) a deadlock with self-controlling path

(b) a dendroid of $P_L$ and $P_C$

Fig. 4 self-controlled deadlock

Now it can be seen that $P_C$ in a simple path has no controlling effect to preserve the token count of D. Thus it is necessary that there exists at least one dendroid for D not to be unmarked. Consider the circuit $C_R$ of a dendroid R of D. If the following inequality,

$$\sum_{P_L \varepsilon C_R} \sum M_0(P_L) > \sum_{P_C \varepsilon C_R} \sum M_0(P_C) \qquad (5)$$

is satisfied at an initial marking $M_0$, then it is satisfied at any $M \varepsilon R(M_0)$. Conversely, if the inequality (5) does not hold at $M_0$, then it does not hold at any $M \varepsilon R(M_0)$ and moreover it can be shown that there exists some M at which left-hand of (5) equals to zero while tokens of other part of D unchanged. At the same time all tokens of $P_L$'s on the feeler of dendroid and all tokens of directed paths in D which connect to any $P_L$ are removed from D. From the discussion above, we obtain our main result.

Theorem 2. Suppose that EMG satisfy Assumption 1 and 2. EMG is live at $M_0$ iff the following conditions are satisfied.

(1) There exists no unmarked deadlock at $M_0$.

(2) Foe each minimal deadlock D, at least one of the following conditions is satisfied.

(i) D satisfies the trap-condition at $M_0$.

(ii) There exists at least one dendroid R of D and the following inequality holds on the circuit $C_R$ of R at $M_0$.

$$\sum_{P_L \in C_R} \Sigma M_0(P_L) > \sum_{P_C \in C_R} \Sigma M_0(P_C)$$

<u>Proof:</u> The necessity of condition (1) is obvious. Suppose EMG is live and condition (2) is not satisfied for a deadlock D. The only possible structure to preserve the tokens on D is a dendroid R. But R does not satisfy (5) that makes D unmarked. By Theorem 1,this leads to a contradiction. Sufficient part is also obvious from the discussion above.

## 4. CONCLUSION

Extended Marked Graphs are introduced to enforce the modelling ability of marked graphs by allowing to include priority function in concurrent processes being modelled. Applications to traffic signal systems and some sequential control problems are apparent. The necessary and sufficient conditions for the liveness of a class of EMG were derived in terms of the initial token distributions and the net structure. Evaluation of reachability of EMG is currently under investigation.

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